

NOTE ON THE MODIFIED q -BERNSTEIN POLYNOMIALS

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ABSTRACT. In the present paper, we propose the modified q -Bernstein polynomials of degree n , which are different q -Bernstein polynomials of Phillips(see [4]). From these the modified q -Bernstein polynomials of degree n , we derive some interesting recurrence formulae for the modified q -Bernstein polynomials.

§1. Introduction

Let $C[0, 1]$ denote the set of continuous function on $[0, 1]$. In [7], Bernstein introduced the following well-known linear positive operators.

$$\begin{aligned} B_n(f : x) &= \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x), \end{aligned} \quad (1)$$

for $f \in C[0, 1]$. $B_n(f : x)$ is called the Bernstein operator for f . The Bernstein polynomial of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad (2)$$

for $k, n \in \mathbb{Z}_+$, where $x \in [0, 1]$ and $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. It is easy to show that

$$B_{0,1}(x) = 1 - x, B_{1,1}(x) = x,$$

$$B_{0,2}(x) = (1-x)^2, B_{1,2}(x) = 2x(1-x), B_{2,2}(x) = x^2,$$

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$$B_{0,3}(x) = (1-x)^3, B_{1,3}(x) = 3x(1-x)^2, B_{2,3}(x) = 3x^2(1-x),$$

$$B_{3,3}(x) = x^3, \dots$$

Many researchers have studied the Bernstein polynomials in the area of approximation theory(see[1-8]). For $k \in \mathbb{Z}_+$, it is easy to show that

$$\begin{aligned} \frac{t^k e^{(1-x)t} x^k}{k!} &= \frac{x^k}{k!} \left(t^k \sum_{n=0}^{\infty} \frac{(1-x)^n t^n}{n!} \right) = \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(1-x)^n t^{n+k}}{n!} \\ &= \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(1-x)^n (n+1) \cdots (n+k)}{(n+k)!} t^{n+k} = x^k \sum_{n=k}^{\infty} \frac{(1-x)^{n-k} \binom{n}{k} t^n}{n!} \\ &= \sum_{n=k}^{\infty} \left(x^k (1-x)^{n-k} \binom{n}{k} \right) \frac{t^n}{n!} = \sum_{n=k}^{\infty} B_{k,n}(x) \frac{t^n}{n!}, \end{aligned}$$

and $B_{k,0}(x) = B_{k,1}(x) = \cdots = B_{k,k-1}(x) = 0$. Thus, we obtain the generating function for $B_{k,n}(x)$ as follows:

$$F^{(k)}(t, x) = \frac{x^k e^{(1-x)t} t^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!}, \quad (3)$$

where $k \in \mathbb{Z}_+$ and $x \in [0, 1]$. From (3), we can derive

$$B_{k,n}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k} & \text{if } n \geq k \\ 0 & \text{if } n < k, \end{cases}$$

for $n, k \in \mathbb{Z}_+$.

Let q be regarded as a real number with $0 < q < 1$ and let us define the q -number as follows:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad (\text{see [1-7]}).$$

Note that $\lim_{q \rightarrow 1} [x]_q = x$. In [4], Phillips introduced the q -extension of Bernstein polynomials. Recently, Simsek and Acikgoz have also studied the q -extension of Bernstein type polynomials([5]). Their q -Bernstein type polynomials are given by

$$\begin{aligned} Y_n(k; x : q) &= \binom{n}{k} \frac{(-1)^k k!}{(1-q)^{n-k}} \sum_{m,l=0}^{\infty} \sum_{j=0}^{n-k} \binom{k+l-1}{l} \binom{n-k}{k} \\ &\quad \times \left(\frac{(-1)^j q^{l+j(1-x)} S(m, k) (x \ln q)^m}{m!} \right), \end{aligned}$$

where $S(m, k)$ are the second kind stirling number.

In this paper we consider the q -extension of the generating function of Bernstein polynomials (see Eq.(3)). From these q -extension of generating function for the Bernstein polynomials, we propose the modified q -Bernstein polynomials of degree n , which are different q -Bernstein polynomials of Phillips. By using the properties of the modified q -Bernstein polynomials, we can obtain some interesting recurrence formulae for the modified q -Bernstein polynomials of degree n .

§2. The modified q -Bernstein polynomials

For $q \in \mathbb{R}$ with $0 < q < 1$, let us consider the q -extension of Eq.(3) as follows:

$$\begin{aligned} F_q^{(k)}(t, x) &= \frac{t^k e^{[1-x]_q t} [x]_q^k}{k!} = \frac{[x]_q^k}{k!} \sum_{n=0}^{\infty} \frac{[1-x]_q^n}{n!} t^{n+k} \\ &= [x]_q^k \sum_{n=0}^{\infty} \binom{n+k}{k} [1-x]_q^n \frac{t^{n+k}}{(n+k)!} \\ &= \sum_{n=k}^{\infty} \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \frac{t^n}{n!}, \end{aligned} \quad (4)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$. Note that $\lim_{q \rightarrow 1} F_q^{(k)}(t, x) = F(t, x)$. By (4), we can define the modified q -Bernstein polynomials as follows:

$$F_q^{(k)}(t, x) = \frac{t^k e^{[1-x]_q t} [x]_q^k}{k!} = \sum_{n=k}^{\infty} B_{k,n}(x, q) \frac{t^n}{n!}, \quad (5)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$. By comparing the coefficients on the both sides of (4) and (5), we obtain the following theorem.

Theorem 1. For $k, n \in \mathbb{Z}_+$, $x \in [0, 1]$, we have

$$B_{k,n}(x, q) = \begin{cases} \binom{n}{k} [x]_q^k [1-x]_q^{n-k} & \text{if } n \geq k \\ 0 & \text{if } n < k. \end{cases}$$

For $0 \leq k \leq n$, we have

$$\begin{aligned} &[1-x]_q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q) \\ &= [1-x]_q \binom{n-1}{k} [x]_q^k [1-x]_q^{n-1-k} + [x]_q \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k} \\ &= \binom{n-1}{k} [x]_q^k [1-x]_q^{n-k} + \binom{n-1}{k-1} [x]_q^k [1-x]_q^{n-k} \\ &= \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, \end{aligned}$$

and the derivative of the modified q -Bernstein polynomials of degree n are also polynomials of degree $n - 1$. That is,

$$\begin{aligned}
& \frac{d}{dx} B_{k,n}(x, q) \\
&= \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} \frac{\ln q}{q-1} q^x + \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} \left(-\frac{\ln q}{q-1} \right) q^{1-x} \\
&= \frac{\ln q}{q-1} \left\{ \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} q^x - \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} q^{1-x} \right\} \\
&= n (q^x B_{k-1,n-1}(x, q) - q^{1-x} B_{k,n-1}(x, q)) \frac{\ln q}{q-1}.
\end{aligned}$$

Therefore, we obtain the following recurrence formulae.

Theorem 2 (Recurrence formulae for $B_{k,n}(x, q)$). For $k, n \in \mathbb{Z}_+, x \in [0, 1]$, we have

$$[1-x]_q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k} = B_{k,n}(x, q),$$

and

$$\frac{d}{dx} B_{k,n}(x, q) = n (q^x B_{k-1,n-1}(x, q) - q^{1-x} B_{k,n-1}(x, q)) \frac{\ln q}{q-1}.$$

Let f be a continuous function on $[0, 1]$. Then the modified q -Bernstein operator is defined by

$$B_{n,q}(f : x) = \sum_{j=0}^n f\left(\frac{j}{n}\right) B_{j,n}(x, q), \quad (6)$$

where $0 \leq x \leq 1, n \in \mathbb{Z}_+$. By Theorem 1 and (6), we see that

$$\begin{aligned}
B_{n,q}(f : x) &= B_{n,q}f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \\
&= [x]_q (1 - [1-x]_q [x]_q (q-1))^{n-1},
\end{aligned}$$

where $f(x) = x$. Thus, we have

$$B_{n,q}(f : x) = f([x]_q) (1 + (1-q)[x]_q[1-x]_q)^{n-1}. \quad (7)$$

From Theorem 1, we note that

$$\begin{aligned}
\sum_{k=0}^n B_{k,n}(x, q) &= \sum_{k=0}^n \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \\
&= \sum_{k=0}^n \binom{n}{k} [x]_q^k (1 - q^{1-x} [x]_q)^{n-k} \\
&= (1 + [x]_q[1-x]_q(q-1))^n = B_{n,q}(1 : x).
\end{aligned}$$

The modified q -Bernstein polynomials are symmetric polynomials. That is, by the definition of the modified q -Bernstein polynomials of degree n , we see that

$$B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}.$$

Thus, we have

$$B_{n-k,n}(1-x, q) = \binom{n}{n-k} [1-x]_q^{n-k} [x]_q^k = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}.$$

Therefore, we obtain the following theorem.

Theorem 3. For $k, n \in \mathbb{Z}_+$, $x \in [0, 1]$, we have

$$B_{n-k,n}(1-x, q) = B_{k,n}(x, q),$$

and

$$\sum_{k=0}^n B_{k,n}(x, q) = (1 + [x]_q [1-x]_q (1-q))^n = B_{n,q}(1 : x).$$

For $t \in \mathbb{C}$, $x \in [0, 1]$, and $n \in \mathbb{Z}_+$, we consider

$$\frac{n!}{2\pi i} \oint_C \frac{([x]_q t)^k}{k!} e^{([1-x]_q t)} \frac{dt}{t^{n+1}}, \quad (9)$$

where C is a circle around the origin and integration is in the positive direction. By the definition of the modified q -Bernstein polynomials and Laurent series, we see that

$$\begin{aligned} & \oint_C \frac{([x]_q t)^k}{k!} e^{[1-x]_q t} \frac{dt}{t^{n+1}} \\ &= \sum_{m=0}^{\infty} \oint_C \frac{B_{k,n}(x, q) t^m}{m!} \frac{dt}{t^{n+1}} = \frac{B_{k,n}(x, q)}{n!} 2\pi i. \end{aligned} \quad (10)$$

From (9) and (10), we note that

$$\frac{n!}{2\pi i} \oint_C \frac{([x]_q t)^k}{k!} e^{[1-x]_q t} \frac{dt}{t^{n+1}} = B_{k,n}(x, q). \quad (11)$$

Also, we see that

$$\begin{aligned} & \oint_C \frac{([x]_q t)^k}{k!} e^{[1-x]_q t} \frac{dt}{t^{n+1}} \\ &= \frac{[x]_q^k}{k!} \sum_{m=0}^{\infty} \oint_C t^{m-n-1+k} dt \frac{[1-x]_q^m}{m!} \\ &= \frac{[x]_q^k}{k!} \left(\frac{[1-x]_q^{n-k}}{(n-k)!} \right) 2\pi i \\ &= \frac{[x]_q^k [1-x]_q^{n-k}}{k!(n-k)!} 2\pi i. \end{aligned} \quad (12)$$

From (9) and (12), we have

$$\frac{n!}{2\pi i} \oint_C \frac{([x]_q t)^k}{k!} e^{([1-x]_q t)} \frac{dt}{t^{n+1}} = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}. \quad (13)$$

By (11) and (13), we easily see that

$$B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}. \quad (14)$$

From (14), we derive

$$\begin{aligned} & \left(\frac{n-k}{n} \right) B_{k,n}(x, q) + \left(\frac{k+1}{n} \right) B_{k+1,n}(x, q) \\ &= \frac{n-k}{n} \binom{n}{k} [x]_q^k [1-x]_q^{n-k} + \frac{k+1}{n} \binom{n}{k+1} [x]_q^{k+1} [1-x]_q^{n-k-1} \\ &= \frac{(n-1)!}{k!(n-k-1)!} [x]_q^k [1-x]_q^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} [x]_q^{k+1} [1-x]_q^{n-k-1} \\ &= [1-x]_q B_{k,n-1}(x, q) + [x]_q B_{k,n-1}(x, q) \\ &= B_{k,n-1}(x, q) + [x]_q (1 - q^{1-x}) B_{k,n-1}(x, q) \\ &= B_{k,n-1}(x, q) + (1-q) [x]_q [1-x]_q B_{k,n-1}(x, q). \end{aligned}$$

Therefore, we can write the modified q -Bernstein polynomials as a linear combination of polynomials of higher order as follows:

Theorem 4. For $k \in \mathbb{Z}_+$, $n \in \mathbb{N}$, and $x \in [0, 1]$, we have

$$\begin{aligned} & \left(\frac{n-k}{n} \right) B_{k,n}(x, q) + \left(\frac{k+1}{n} \right) B_{k+1,n}(x, q) \\ &= B_{k,n-1}(x, q) + (1-q) [x]_q [1-x]_q B_{k,n-1}(x, q). \end{aligned}$$

By (14), we easily see that

$$\begin{aligned} & \left(\frac{n-k+1}{k} \right) \left(\frac{[x]_q}{[1-x]_q} \right) B_{k-1,n}(x, q) \\ &= \left(\frac{n-k+1}{k} \right) \left(\frac{[x]_q}{[1-x]_q} \right) \binom{n}{k-1} [x]_q^{k-1} [1-x]_q^{n-k+1} \\ &= \frac{n!}{k!(n-k)!} [x]_q^k [1-x]_q^{n-k} = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}. \end{aligned}$$

Thus, we obtain the following corollary.

Corollary 5. For $n, k \in \mathbb{N}$, and $x \in [0, 1]$, we have

$$\left(\frac{n-k+1}{k}\right) \left(\frac{[x]_q}{[1-x]_q}\right) B_{k-1,n}(x, q) = B_{k,n}(x, q).$$

From the definition of the modified q -Bernstein polynomials and binomial theorem, we note that

$$\begin{aligned} B_{k,n}(x, q) &= \binom{n}{k} [x]_q^k [1-x]_q^{n-k} = \binom{n}{k} [x]_q^k (1 - q^{1-x} [x]_q)^{n-k} \\ &= \binom{n}{k} [x]_q^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l q^{l(1-x)} [x]_q^l \\ &= \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} (-1)^l q^{l(1-x)} [x]_q^{l+k} \\ &= \sum_{l=0}^{n-k} \binom{k+l}{k} \binom{n}{k+l} (-1)^l q^{l(1-x)} [x]_q^{l+k} \\ &= \sum_{i=k}^n \binom{i}{k} \binom{n}{i} (-1)^{i-k} q^{(1-x)(i-k)} [x]_q^i. \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 6. For $k, n \in \mathbb{Z}_+$, and $x \in [0, 1]$, we have

$$B_{k,n}(x, q) = \sum_{i=k}^n \binom{i}{k} \binom{n}{i} (-1)^{i-k} q^{(1-x)(i-k)} [x]_q^i.$$

It is possible to write each power basis element of $[x]_q^k$, in the linear combination of the modified q -Bernstein polynomials by using the degree evaluation formulae and induction method in mathematics. From the property of the modified q -Bernstein polynomials, we easily see that

$$\begin{aligned} \sum_{k=0}^n \frac{k}{n} B_{k,n}(x, q) &= \sum_{k=0}^n \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \\ &= \sum_{k=1}^n \binom{n-1}{k-1} [x]_q^k [1-x]_q^{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_q^{k+1} [1-x]_q^{n-1-k} \\ &= [x]_q ([x]_q + [1-x]_q)^{n-1}, \end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^n \frac{\binom{k}{2}}{\binom{n}{2}} B_{k,n}(x, q) &= \sum_{k=1}^n \frac{k(k-1)}{n(n-1)} \binom{n}{k} [1-x]_q^{n-k} [x]_q^k \\
&= \sum_{k=2}^{\infty} \frac{k(k-1)}{n(n-1)} \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \\
&= \sum_{k=2}^n \binom{n-2}{k-2} [x]_q^k [1-x]_q^{n-k} \\
&= \sum_{k=0}^{n-2} \binom{n-2}{k} [x]_q^{k+2} [1-x]_q^{n-2-k} \\
&= [x]_q^2 ([x]_q + [1-x]_q)^{n-2}.
\end{aligned}$$

Continuing this process, we obtain

$$\sum_{k=i-1}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = [x]_q^i ([x]_q + [1-x]_q)^{n-i},$$

for $i \in \mathbb{N}$. Therefore we obtain the following theorem.

Theorem 7. For $n \in \mathbb{Z}_+$, $i \in \mathbb{N}$ and $x \in [0, 1]$, we have

$$\frac{1}{([1-x]_q + [x]_q)^{n-i}} \sum_{k=i-1}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = [x]_q^i.$$

The Bernoulli polynomials of order $k \in \mathbb{N}$ are defined as

$$\left(\frac{t}{e^t - 1} \right)^k e^{xt} = \underbrace{\left(\frac{t}{e^t - 1} \right)^k}_{k\text{-times}} \times \cdots \times \left(\frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (15)$$

and $B_n^{(k)} = B_n^{(k)}(0)$ are called the n -th Bernoulli numbers of order k . It is well known that the second kind stirling numbers are defined as

$$\frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!}, \quad (16)$$

for $k \in \mathbb{N}$. From (5), we note that

$$\begin{aligned}
\frac{([x]_q t)^k e^{[1-x]_q t}}{k!} &= \frac{[x]_q^k (e^t - 1)^k}{k!} \left(\frac{t}{e^t - 1} \right)^k e^{[1-x]_q t} \\
&= [x]_q^k \left(\sum_{m=0}^{\infty} S(m, k) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} B_n^{(k)} ([1-x]_q) \frac{t^n}{n!} \right) \\
&= [x]_q^k \sum_{l=0, m+n=l}^{\infty} \left(\sum_{n=0}^l \frac{B_n^{(k)} ([1-x]_q) S(l-n, k) \binom{l}{n}}{n! (l-n)!} \right) \frac{t^l}{l!}.
\end{aligned} \tag{17}$$

By (5) and (17), we have

$$B_{k,l}(x, q) = [x]_q^k \sum_{n=0}^l B_n^{(k)} ([1-x]_q) S(l-n, k) \binom{l}{n},$$

and $B_{k,0}(x, q) = B_{k,1}(x, q) = \cdots = B_{k,k-1}(x, q) = 0$, where $B_n^{(k)} ([1-x]_q)$ are called the n -th Bernoulli polynomials of order k .

Let Δ be the shift difference operator with $\Delta f(x) = f(x+1) - f(x)$. By iterative method, we easily see that

$$\Delta^n f(0) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(k), \tag{18}$$

for $n \in \mathbb{N}$. From (16) and (18), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!} &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{lt} \\
&= \sum_{n=0}^{\infty} \left\{ \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} l^n \right\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Delta^k 0^n}{k!} \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on the both sides, we have

$$S(n, k) = \frac{\Delta^k 0^n}{k!}, \tag{19}$$

for $n, k \in \mathbb{Z}_+$. Thus, we note that

$$B_{k,l}(x, q) = [x]_q^k \sum_{n=0}^l B_n^{(k)} ([1-x]_q) \binom{l}{n} \frac{\Delta^k 0^{l-n}}{k!}. \tag{20}$$

Let $(Eh)(x) = h(x+1)$ be the shift operator. Then the q -difference operator is defined by

$$\Delta_q^n = \Pi_{i=0}^{n-1} (E - q^i I), \quad (\text{see [2]}),$$

where I is an identity operator. For $f \in C[0, 1]$ and $n \in \mathbb{N}$, we have

$$\Delta_q^n f(0) = \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{\binom{n}{2}} f(n-k),$$

where $\binom{n}{k}_q$ is Gaussian binomial coefficient.

Let $F_q(t)$ be the generating function of the q -extension of the second kind stirling number as follows:

$$\begin{aligned} F_q(t) &= \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j}_q q^{\binom{k-j}{2}} e^{[j]_q t} \\ &= \sum_{n=0}^{\infty} S(n, k : q) \frac{t^n}{n!}, \quad (\text{see [2]}). \end{aligned} \quad (21)$$

From (21), we have

$$\begin{aligned} S(n, k : q) &= \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k-j]_q^n \\ &= \frac{q^{-\binom{k}{2}}}{[k]_q!} \Delta_q^k 0^n, \end{aligned} \quad (22)$$

where $[k]_q! = [k]_q [k-1]_q \cdots [2]_q [1]_q$. It is not difficult to show that

$$[x]_q^n = \sum_{k=0}^n q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(n, k : q), \quad (\text{see [2]}), \quad (23)$$

Thus, we have

$$\sum_{k=0}^i q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(i, k : q) = \frac{1}{([1-x]_q + [x]_q)^{n-i}} \sum_{k=i-1}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q). \quad (24)$$

Therefore, we obtain the following theorem.

Theorem 8. For $n \in \mathbb{Z}_+$, $i \in \mathbb{N}$ and $x \in [0, 1]$, we have

$$\frac{1}{([1-x]_q + [x]_q)^{n-i}} \sum_{k=i-1}^n \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = \sum_{k=0}^i q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(i, k : q),$$

where $\binom{x}{k}_q = \frac{[x]_q [x-1]_q \cdots [x-k+1]_q}{[k]_q!}$.

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